Journal Title
XX(X):1–14

Reprints and permission:
sagepub.co.uk/journalsPermissions.nav
DOI: 10.1177/ToBeAssigned
www.sagepub.com/
SAGE

Supplemental material: Designing clinical trials for the comparison of single and multiple quantiles with right-censored data

Anonymous author[1]

Contents

[1] Anonymous university

Corresponding author:
Anonymous, Anonymous Laboratory
Email: anonymous email

1 Proofs and Technical details

1.1 Univariate test

In this section, we present the proofs for the results in Section 2.1 of the main document, for the univariate test. We first recall the expression of the statistical test in the univariate case.

$$\mathcal{T}_n = \sqrt{n}\frac{\hat{F}_1^{-1}(p) - \hat{F}_2^{-1}(p)}{\hat{\sigma}}, \text{ with}$$
$$\hat{\sigma}^2 = (1-p)^2\left(\frac{\hat{\phi}_1}{\hat{\mu}_1\hat{f}_1(\hat{F}_1^{-1}(p))^2} + \frac{\hat{\phi}_2}{\hat{\mu}_2\hat{f}_2(\hat{F}_2^{-1}(p))^2}\right).$$

Result 1. *Under* $\mathcal{H}_0 : F_1^{-1}(p) = F_2^{-1}(p)$, $\mathcal{T}_n \xrightarrow{d} \mathcal{N}(0,1)$ *as* $n \to \infty$

Proof. From Lemma 1 from Kosorok[2], for $k = 1, 2$,

$$\hat{F}_k^{-1}(p) - F_k^{-1}(p) = \frac{\hat{F}_k(F_k^{-1}(p)) - p}{f_k(F_k^{-1}(p))} + o_p\left(\frac{1}{\sqrt{n_k}}\right)$$

It follows from Theorem 6.3.1 from Fleming and Harrington[3] that:

$$\sqrt{n_k}\left(\frac{\hat{F}_k(F_k^{-1}(p)) - p}{f_k(F_k^{-1}(p))}\right) \xrightarrow{d} \mathcal{N}\left(0, (1-p)^2\frac{\phi_k}{f_k(F_k^{-1}(p))}\right), \text{ with}$$

$$\phi_k = \int_0^{F_k^{-1}(p)} \frac{d\Lambda_k(x)}{H_k(x)}.$$

From the independence between treatment groups, as $n \to \infty$, we have that

$$\sqrt{n}\left[\frac{\hat{F}_1(F_1^{-1}(p)) - p}{f_1(F_1^{-1}(p))} - \frac{\hat{F}_2(F_2^{-1}(p)) - p}{f_2(F_2^{-1}(p))}\right] \xrightarrow{d} \mathcal{N}\left(0,\ (1-p)^2\left[\frac{\phi_1}{\mu_1 f_1(F_1^{-1}(p))^2} + \frac{\phi_2}{\mu_2 f_2(F_2^{-1}(p))^2}\right]\right).$$

Then, $\hat{\sigma} \to \sigma$ as $n \to \infty$ from the consistency of Greenwood's estimator of the Kaplan-Meier variance, the consistency of the density estimators and the consistency of $\hat{\mu}_k$, $k = 1, 2$. Finally under $\mathcal{H}_0$, we have the decomposition

$$\hat{F}_1^{-1}(p) - \hat{F}_2^{-1}(p) = \hat{F}_1^{-1}(p) - F_1^{-1}(p) - (\hat{F}_2^{-1}(p) - F_2^{-1}(p)),$$

which concludes the proof.

Result 2. *Under* $\mathcal{H}_1 : F_1^{-1}(p) - F_2^{-1}(p) = \Delta$, $\mathcal{T}_n - \sqrt{n}\frac{\Delta}{\hat{\sigma}} \xrightarrow{d} \mathcal{N}(0,1)$ *as* $n \to \infty$

Proof. Under $\mathcal{H}_1$,

$$\sqrt{n}(\hat{F}_1^{-1}(p) - \hat{F}_2^{-1}(p)) = \sqrt{n}(\hat{F}_1^{-1}(p) - F_1^{-1}(p)) - \sqrt{n}(\hat{F}_2^{-1}(p) - F_2^{-1}(p)) + \sqrt{n}\Delta.$$

We denote

$$Z_n = \sqrt{n}(\hat{F}_1^{-1}(p) - F_1^{-1}(p)) - \sqrt{n}(\hat{F}_2^{-1}(p) - F_2^{-1}(p)).$$

Then, from Lemma 1 from Kosorok[2],

$$Z_n = \sqrt{n}\left(\frac{\hat{F}_1(F_1^{-1}(p)) - p}{f_1(F_1^{-1}(p))}\right) - \sqrt{n}\left(\frac{\hat{F}_2(F_2^{-1}(p)) - p}{f_2(F_2^{-1}(p))}\right) + o_p\left(\frac{1}{\sqrt{n}}\right).$$

It follows, for $n \to \infty$:

$$Z_n \xrightarrow{d} \mathcal{N}\left(0, \frac{(1-p)^2\phi_1}{\mu_1 f_1(F_1^{-1}(p))^2} + \frac{(1-p)^2\phi_2}{\mu_2 f_2(F_2^{-1}(p))^2}\right)$$

From the consistency of $\hat{\sigma}$, we therefore have $Z_n/\hat{\sigma} \xrightarrow{d} \mathcal{N}(0,1)$, which gives the desired result using the relation

$$\mathcal{T}_n = Z_n/\hat{\sigma} + \sqrt{n}\Delta/\hat{\sigma}.$$

Result 3. *We have the following asymptotic formula for the power of the test of level α:*

$$1 - \beta \approx 1 - \Phi\left(q_{1-\alpha/2} - \frac{\sqrt{n}}{\hat{\sigma}}\Delta\right) + \Phi\left(-q_{1-\alpha/2} - \frac{\sqrt{n}}{\hat{\sigma}}\Delta\right)$$

where Φ is the cumulative distribution function of the standard normal variable and $q_{1-\alpha/2}$ is the quantile of level $1-\alpha/2$ of the standard normal distribution.

Proof. Using the derived expression for the test statistic and denoting as $q_{1-\frac{\alpha}{2}}$ the quantile of order $1-\frac{\alpha}{2}$ of the standard normal distribution,

$$\begin{aligned}
1 - \beta &= P_{\mathcal{H}_1}(|\mathcal{T}_n| > q_{1-\frac{\alpha}{2}}) \\
&= P_{\mathcal{H}_1}\left(\sqrt{n}\left|\frac{\hat{F}_1^{-1}(p) - \hat{F}_2^{-1}(p)}{\hat{\sigma}}\right| > q_{1-\frac{\alpha}{2}}\right) \\
&= P_{\mathcal{H}_1}\left(\sqrt{n}\left|\left(\frac{\hat{F}_1^{-1}(p) - F_1^{-1}(p)}{\hat{\sigma}}\right) - \left(\frac{\hat{F}_2^{-1}(p) - F_2^{-1}(p)}{\hat{\sigma}}\right) + \frac{\Delta}{\hat{\sigma}}\right| > q_{1-\alpha/2}\right) \\
&\approx P_{\mathcal{H}_1}\left(\frac{\sqrt{n}}{\hat{\sigma}}\left(\frac{\hat{F}_1(F_1^{-1}(p)) - p}{f_1(F_1^{-1}(p))} - \frac{\hat{F}_2(F_2^{-1}(p)) - p}{f_2(F_2^{-1}(p))}\right) > q_{1-\alpha/2} - \frac{\sqrt{n}\Delta}{\hat{\sigma}}\right) \\
&+ P_{\mathcal{H}_1}\left(\frac{\sqrt{n}}{\hat{\sigma}}\left(\frac{\hat{F}_1(F_1^{-1}(p)) - p}{f_1(F_1^{-1}(p))} - \frac{\hat{F}_2(F_2^{-1}(p)) - p}{f_2(F_2^{-1}(p))}\right) < -q_{1-\alpha/2} - \frac{\sqrt{n}\Delta}{\hat{\sigma}}\right) \\
&\approx P_{\mathcal{H}_1}\left(\frac{Z_n}{\hat{\sigma}} > q_{1-\alpha/2} - \frac{\sqrt{n}\Delta}{\hat{\sigma}}\right) + P_{\mathcal{H}_1}\left(\frac{Z_n}{\hat{\sigma}} < -q_{1-\alpha/2} - \frac{\sqrt{n}\Delta}{\hat{\sigma}}\right),
\end{aligned}$$

where the expression in the fourth equality comes from Lemma 1 from Kosorok[2]. We conclude from the convergence of $Z_n/\hat{\sigma}$ towards a centered Gaussian random variable.

1.2 Multivariate test

In this section, we present the proofs for the results in Section 2.2 of the main document, for the multivariate test.

Result 4. *For $k = 1, 2$,*

$$\sqrt{n}\begin{bmatrix}\hat{F}_k^{-1}(p_1) - F_k^{-1}(p_1)\\ \vdots \\ \hat{F}_k^{-1}(p_J) - F_k^{-1}(p_J)\end{bmatrix} \xrightarrow{d} \mathcal{N}(0, \Upsilon_{F_k}) \text{ as } n \to \infty,$$

where:

$$(\Upsilon_{F_k})_{jl} = \begin{cases} \dfrac{(1-p_j)^2 \int_0^{F_k^{-1}(p_j)} \dfrac{d\Lambda_k(x)}{H_k(x)}}{\mu_k (f_k(F_k^{-1}(p_j)))^2}, \text{ if } j = l \\ \\ \dfrac{(1-p_j)(1-p_k) \int_0^{F_k^{-1}(p_j)\wedge F_k^{-1}(p_l)} \dfrac{d\Lambda_k(x)}{H_k(x)}}{\mu_k f_k(F_k^{-1}(p_j)) f_k(F_k^{-1}(p_l))}, \text{ otherwise.}\end{cases}$$

Proof. From Lemma 1 in Kosorok[2], we have:

$$\begin{bmatrix}\hat{F}_k^{-1}(p_1) - F_k^{-1}(p_1)\\ \vdots \\ \hat{F}_k^{-1}(p_J) - F_k^{-1}(p_J)\end{bmatrix} = U_k + o_p(1/\sqrt{n_k}),$$

where

$$U_k = \begin{bmatrix}\dfrac{\hat{F}_k(F_k^{-1}(p_1)) - p_1}{f_k(F_k^{-1}(p_1))}\\ \vdots \\ \dfrac{\hat{F}_k(F_k^{-1}(p_J)) - p_J}{f_k(F_k^{-1}(p_J))}\end{bmatrix}.$$

From Theorem 6.3.1 from Fleming and Harrington[3], $\sqrt{n_k}U_k$ converges to a centered multivariate Gaussian random variable where its variance matrix has entries (j, l) equal to:

$$\frac{(1-p_j)(1-p_l) \int_0^{F_k^{-1}(p_j)\wedge F_k^{-1}(p_l)} \dfrac{d\Lambda_k(x)}{H_k(x)}}{f_k(F_k^{-1}(p_j)) f_k(F_k^{-1}(p_l))}.$$

The result follows since $n_k/n \to \mu_k$ as $n \to \infty$.

Result 5. *Under* $\mathcal{H}_0 : F_1^{-1}(p_j) = F_2^{-1}(p_j), j = 1, ..., J,$

$$\mathcal{Z}_n = \sqrt{n} \begin{bmatrix} \hat{F}_1^{-1}(p_1) - \hat{F}_2^{-1}(p_1) \\ \vdots \\ \hat{F}_1^{-1}(p_J) - \hat{F}_2^{-1}(p_J) \end{bmatrix} \xrightarrow{d} \mathcal{N}(0, \Psi) \text{ as } n \to \infty,$$

where $\Psi = \Upsilon_{F_1} + \Upsilon_{F_2}$.

The test statistic, expressed as $\mathfrak{T}_n = \mathcal{Z}_n^T \Psi^{-1} \mathcal{Z}_n$, *converges in distribution towards a* χ^2_J *as* $n \to \infty$ *under* $\mathcal{H}_0$.

Proof. We have under $\mathcal{H}_0$,

$$\mathcal{Z}_n = \sqrt{n} \begin{bmatrix} \hat{F}_1^{-1}(p_1) - F_1^{-1}(p_1) \\ \vdots \\ \hat{F}_1^{-1}(p_J) - F_1^{-1}(p_J) \end{bmatrix} - \sqrt{n} \begin{bmatrix} \hat{F}_2^{-1}(p_1) - F_2^{-1}(p_1) \\ \vdots \\ \hat{F}_2^{-1}(p_J) - F_2^{-1}(p_J) \end{bmatrix}$$

Since the two treatment groups are independent, we have from Result 4 that, as $n \to \infty$, $\mathcal{Z}_n \xrightarrow{d} \mathcal{N}(0, \Psi)$, where $\Psi = \Upsilon_{F_1} + \Upsilon_{F_2}$. From these results we conclude that under $\mathcal{H}_0$, $\mathfrak{T}_n \xrightarrow{d} \chi^2_J$ as $n \to \infty$.

Result 6. *Under* $\mathcal{H}_1 : F_1^{-1}(p_j) - F_2^{-1}(p_j) = \Delta_j, j = 1, ..., J,$ $\mathfrak{T}_n$ *is asymptotically equivalent to* $\chi^2_J(\Psi^{-1/2}\xi)$, *an uncentered chi-squared distribution with* J *degrees of freedom and mean* $\Psi^{-1/2}\xi$, *with*

$$\xi = \sqrt{n} \begin{bmatrix} \Delta_1 \\ \vdots \\ \Delta_J \end{bmatrix}.$$

We then have the following asymptotic formula for the power of the test of level α*:*

$$1 - \beta = F_{\chi^2_J(\Psi^{-1/2}\xi)}(q_{J,1-\alpha}),$$

denoting as $F_{\chi^2_J(\Psi^{-1/2}\xi)}$ *the cumulative distribution function of this chi-squared distribution and* $q_{J,1-\alpha}$ *as the quantile of order* $1 - \alpha$ *of the chi-squared distribution with* J *degrees of freedom.*

Proof. We denote:

$$Y_n = \sqrt{n} \begin{bmatrix} \hat{F}_1^{-1}(p_1) - F_1^{-1}(p_1) \\ \vdots \\ \hat{F}_1^{-1}(p_J) - F_1^{-1}(p_J) \end{bmatrix} - \sqrt{n} \begin{bmatrix} \hat{F}_2^{-1}(p_1) - F_2^{-1}(p_1) \\ \vdots \\ \hat{F}_2^{-1}(p_J) - F_2^{-1}(p_J) \end{bmatrix}.$$

Under $\mathcal{H}_1$, we write:

$$\mathcal{Z}_n = Y_n + \xi.$$

From Lemma 1 in Kosorok[2], we have:

$$Y_n = \sqrt{n}\begin{bmatrix} \dfrac{\hat{F}_1(F_1^{-1}(p_1)) - p_1}{f_1(F_1^{-1}(p_1))} \\ \vdots \\ \dfrac{\hat{F}_1(F_1^{-1}(p_J)) - p_J}{f_1(F_1^{-1}(p_J))} \end{bmatrix} - \sqrt{n}\begin{bmatrix} \dfrac{\hat{F}_2(F_2^{-1}(p_1)) - p_1}{f_2(F_2^{-1}(p_1))} \\ \vdots \\ \dfrac{\hat{F}_2(F_2^{-1}(p_J)) - p_J}{f_2(F_2^{-1}(p_J))} \end{bmatrix} + o_p(1/\sqrt{n}).$$

The first and second term on the right-hand side of this equation converge in distribution respectively to $\mathcal{N}(0, \Upsilon_{F_1})$ and $\mathcal{N}(0, \Upsilon_{F_2})$ as $n \to \infty$. Therefore,

$$Y_n \xrightarrow{d} \mathcal{N}(0, \Psi), \text{ as } n \to \infty,$$

and $\mathcal{Z}_n$ is asymptotically equivalent to a $\mathcal{N}(\xi, \Psi)$. From the consistency of $\hat{\Psi}$, $\mathcal{Z}_n^T \hat{\Psi}^{-1} \mathcal{Z}_n$ is asymptotically equivalent to $\chi^2_J(\Psi^{-1/2}\xi)$. We conclude from the definition of the power,

$$\begin{aligned} 1 - \beta &= P_{\mathcal{H}_1}(\mathfrak{T}_n > q_{J,1-\alpha}) \\ &\approx F_{\chi^2_J(\Psi^{-1/2}\xi)}(q_{J,1-\alpha}). \end{aligned}$$

2 Details on the estimation of the density

In the main document, two different methods are proposed for the estimation of $f_k(F_k^{-1}(p))$, $k = 1, 2$: a resampling procedure based on the method from Lin *et al.*[4] and a kernel density estimator. The first method only estimates the densities at one point, the quantile $F_k^{-1}(p)$, while the second method estimates the whole function $f_k(t)$ from which the estimator at the quantile is obtained by setting $t = F_k^{-1}(p)$. In Section 2.1 the resampling procedure is explained, while details for the kernel density estimator are provided in Section 2.2.

2.1 Resampling procedure

We propose a resampling procedure that allows to estimate $f(F^{-1}(p))$ the density at a given quantile, for a given probability $0 < p < 1$. This method, inspired by Lin *et al.*[4] and presented in Farah *et al.*[1], consists of generating multiple samples of the zero-mean Gaussian variable from which a least square estimator is constructed. We require the densities at the quantiles to be strictly positive, and we denote as $\hat{F}$ the consistent estimator for F obtained from the usual Kaplan-Meier estimation. Taking this estimator we obtain $\hat{F}^{-1}(p)$ the estimators of the inverse distribution at p. Then we propose the following resampling procedure:

1. Generate B realizations of the Gaussian $\varepsilon \sim \mathcal{N}(0, \sigma_\varepsilon^2)$, denoted by $\varepsilon_1, ..., \varepsilon_B$

2. Calculate $\sqrt{n}\left(\hat{F}\left(\hat{F}^{-1}(p)+\frac{\varepsilon_b}{\sqrt{n}}\right)-p\right), b=1,...,B$ and denote them as y_b, then the least squares estimate of $f(F^{-1}(p))$ is $\hat{A}=(x'x)^{-1}x'Y$, where $x=(\varepsilon_1,...,\varepsilon_B)^T$ and $Y=(y_1,...,y_B)^T$

We advocate that the variance of the Gaussian variables must be carefully chosen as it may impact the quality of estimation of the density. A grid-search algorithm is used for automatic variance selection in practical applications. Details about the resampling-based procedure, variance selection and theoretical results on the consistency of the estimator are provided in Farah *et al.*[1].

2.2 Kernel density estimation

In the presence of censoring, a typical kernel estimator of the density can be constructed by estimating the censoring distribution from the Kaplan-Meier estimator. In Földes *et al.*[5], Diehl and Stute[6] the following estimator has been proposed:

$$\hat{f}_h(t)=\frac{1}{nh}\sum_{i=1}^{n}\frac{\delta_i}{\hat{S}_{\text{cens}}(T_i)}K\left(\frac{T_i-t}{h}\right),$$

where $\hat{S}_{\text{cens}}$ is the Kaplan-Meier estimator of the censoring survival function and K a kernel satisfying standard conditions. In order to compute this estimator, the kernel and the bandwidth must be chosen. In our implementation, a Gaussian kernel was taken and the bandwidth was obtained from cross-validation. For the choice of the bandwidth, the goal of the method is to try to minimize the Integrated Squared Error (ISE), defined, for $k=1,2$, as:

$$\begin{aligned}\text{ISE}(\hat{f}_h)&=\int\left(\hat{f}_h(t)-f_k(t)\right)^2dt\\&=\int\hat{f}_h^2(t)dt-2\int\hat{f}_h(t)f_k(t)dt+\int f_k^2(t)dt.\end{aligned}$$

In this expression the last term does not depend on h and can be omitted. On the other hand, the term $\int f_h(t)f_k(t)dt$ is estimated by:

$$\hat{J}(h)=\frac{1}{n(n-1)h}\sum_{i\neq j}K\left(\frac{T_i-T_j}{h}\right)\frac{\delta_{ik}\delta_{jk}}{\hat{S}_{\text{cens}}(T_i)\hat{S}_{\text{cens}}(T_j)}.$$

Using the consistency of the Kaplan-Meier estimator, it can easily be shown that this estimator converges in probability, as n tends to infinity, towards $\int\hat{f}_h(t)f_k(t)dt$ (see Marron and Padgett[7]). In conclusion, our cross-validated estimator is defined as the Gaussian kernel estimator with bandwidth chosen as the minimizer of $\int\hat{f}_h^2(t)dt-2\hat{J}(h)$, that is:

$$\hat{h}=\arg\min_h\left\{\int\hat{f}_h^2(t)dt-\frac{2}{n(n-1)h}\sum_{i\neq j}K\left(\frac{T_i-T_j}{h}\right)\frac{\delta_{ik}\delta_{jk}}{\hat{S}_{\text{cens}}(T_i)\hat{S}_{\text{cens}}(T_j)}\right\}.$$

We refer the reader to Marron and Padgett[7] for more details about the cross-validation bandwidth selector and theoretical results regarding its validity.

3 Additional simulations

In this section, we present additional simulations for the Section 3.1 of the main document, in the context of planning clinical trials. We report the analytical power and type I error, as well as the empirical rejection rate obtained through ten thousand simulations. We recall the simulation scenarios, in which the survival time in the control arm follows an exponential distribution with rate λ_a, and in the experimental arm it is specified as follows:

- Scenario 1 (proportional hazards): Exponential with rate λ_b.
- Scenario 2 (late differences): Piecewise exponential with rate λ_a until time t_{cut} and λ_b onward.

In all scenarios, the distribution of censoring time is exponential with rate λ_{cens}. Using the expression derived in the previous section, it is possible to compute the analytical power as a function of the parameters for each scenario.

Mean computational time per iteration is reported in seconds. All simulations were conducted on a workstation with an Intel Core i5-13600H CPU and 32 GB of RAM. Computation times were measured using `system.time` from `R` and reported as the elapsed wall-clock time. The code required to reproduce the results reported here was implemented in R and is publicly available at `https://github.com/beafarah/dens-estimation-at-quantile/`.

3.1 Test at $p = 0.5$*, 25% of censoring*

In this scenario, we set $\lambda_a = 1.5$, $\lambda_{\text{cens}} = 0.48$, $t_{\text{cut}} = 0.2$.

Δ	**Scenario 1**			**Scenario 2**		
	Empirical	**Formula**	**Comp. time**	**Empirical**	**Formula**	**Comp. time**
			Sample size $n_i = 50$			
0	0.0103	0.05	$1.3 \times 10^{-2}(8.46 \times 10^{-3})$	0.0015	0.05	$9.16 \times 10^{-3}(9.41 \times 10^{-3})$
0.1	0.0504	0.1236	$1.01 \times 10^{-2}(8.88 \times 10^{-3})$	0.0174	0.1153	$1.28 \times 10^{-2}(1.54 \times 10^{-2})$
0.2	0.3507	0.4147	$1.19 \times 10^{-2}(9.81 \times 10^{-3})$	0.3908	0.4682	$1.36 \times 10^{-2}(1.52 \times 10^{-2})$
			Sample size $n_i = 100$			
0	0.0257	0.05	$1.3 \times 10^{-2}(1.01 \times 10^{-2})$	0.0060	0.05	$9.5 \times 10^{-3}(8.45 \times 10^{-3})$
0.1	0.1583	0.2000	$1.25 \times 10^{-2}(1.11 \times 10^{-2})$	0.0991	0.1831	$1.34 \times 10^{-2}(2.08 \times 10^{-2})$
0.2	0.7280	0.6939	$1.35 \times 10^{-2}(8.45 \times 10^{-3})$	0.8645	0.7577	$1.49 \times 10^{-2}(2.24 \times 10^{-2})$
			Sample size $n_i = 500$			
0	0.047	0.05	$1.34 \times 10^{-2}(1.05 \times 10^{-2})$	0.047	0.05	$1.26 \times 10^{-2}(1.04 \times 10^{-2})$
0.1	0.714	0.703	$1.75 \times 10^{-2}(1.83 \times 10^{-2})$	0.782	0.766	$1.45 \times 10^{-2}(1.26 \times 10^{-2})$
0.2	1.000	1.000	$1.57 \times 10^{-2}(1.48 \times 10^{-2})$	1.000	1.000	$1.78 \times 10^{-2}(1.73 \times 10^{-2})$

Table 1. Type I error and power of the test of equality of quantiles for sample sizes $n_i = 50, 100, 500$, at $p = 0.5$. Mean computational time and standard error per iteration are shown in seconds.

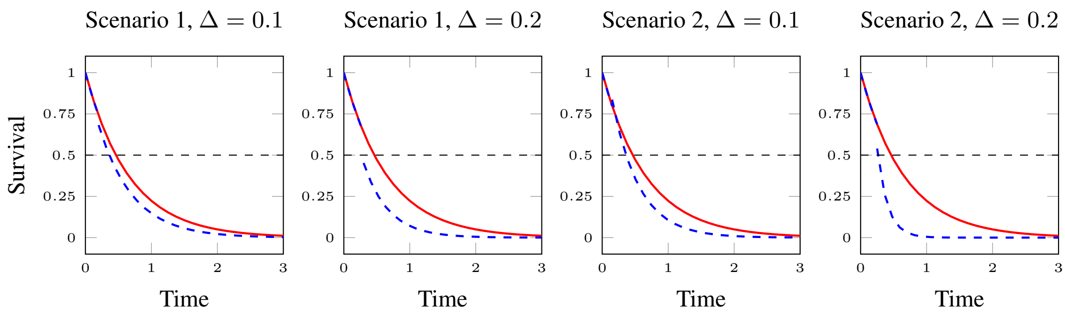

Figure 1. True survival curves for the test at $p = 0.5$ with 25% of censoring.

3.2 Test at $p = 0.75$, 25% of censoring

We illustrate the test at a late quantile, closer to the tail of the distributions. We set $\lambda_a = 1.5$, $\lambda_{\text{cens}} = 0.48$, $t_{\text{cut}} = 0.2$.

Δ	Scenario 1			Scenario 2		
	Empirical	**Formula**	**Comp. time**	**Empirical**	**Formula**	**Comp. time**
			Sample size $n_i = 50$			
0	0.0040	0.05	$9.4 \times 10^{-3}(1.18 \times 10^{-2})$	0.0051	0.05	$1.31 \times 10^{-2}(1.60 \times 10^{-2})$
0.1	0.0086	0.0685	$1.15 \times 10^{-2}(1.36 \times 10^{-2})$	0.0092	0.0669	$1.56 \times 10^{-2}(2.23 \times 10^{-2})$
0.2	0.0361	0.1356	$1.44 \times 10^{-2}(2.07 \times 10^{-2})$	0.0306	0.1314	$1.43 \times 10^{-2}(1.55 \times 10^{-2})$
			Sample size $n_i = 100$			
0	0.0109	0.05	$1.3 \times 10^{-2}(2.01 \times 10^{-2})$	0.0108	0.05	$1.44 \times 10^{-2}(1.62 \times 10^{-2})$
0.1	0.0330	0.0874	$1.42 \times 10^{-2}(1.92 \times 10^{-2})$	0.0312	0.0841	$1.44 \times 10^{-2}(1.45 \times 10^{-2})$
0.2	0.1440	0.2243	$1.54 \times 10^{-2}(1.99 \times 10^{-2})$	0.1365	0.2158	$1.51 \times 10^{-2}(1.69 \times 10^{-2})$
			Sample size $n_i = 500$			
0	0.0355	0.05	$1.82 \times 10^{-2}(1.64 \times 10^{-3})$	0.0336	0.05	$1.82 \times 10^{-2}(2.41 \times 10^{-2})$
0.1	0.2288	0.2444	$1.68 \times 10^{-2}(2.13 \times 10^{-2})$	0.2335	0.2270	$1.63 \times 10^{-2}(1.58 \times 10^{-2})$
0.2	0.7779	0.7650	$1.88 \times 10^{-2}(2.04 \times 10^{-2})$	0.8055	0.7447	$1.81 \times 10^{-2}(2.02 \times 10^{-2})$

Table 2. Type I error and power of the test of equality of quantiles for sample sizes $n_i = 50, 100, 500$, at $p = 0.75$.

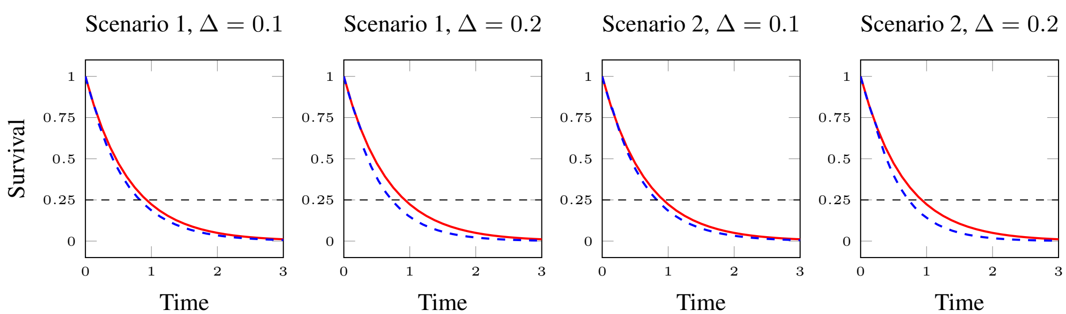

Figure 2. True survival curves for the test at $p = 0.75$ with 25% of censoring.

3.3 Test at $p = 0.9$, 5% of censoring

We illustrate the test at a high extreme quantile, at the tail of the distributions. We set $\lambda_a = 3$, $\lambda_{\text{cens}} = 0.1$, $t_{\text{cut}} = 0.2$.

Δ	Scenario 1			Scenario 2		
	Empirical	**Formula**	**Comp. time**	**Empirical**	**Formula**	**Comp. time**
Sample size $n_i = 50$						
0	0.0097	0.05	$1.61 \times 10^{-2}(1.39 \times 10^{-2})$	0.0056	0.05	$1.18 \times 10^{-2}(1.343 \times 10^{-2})$
0.1	0.0232	0.081	$2.05 \times 10^{-2}(2.76 \times 10^{-2})$	0.0147	0.0832	$1.12 \times 10^{-2}(1.289 \times 10^{-2})$
0.2	0.102	0.1991	$1.16 \times 10^{-2}(2.05 \times 10^{-2})$	0.0709	0.2126	$1.35 \times 10^{-2}(2.06 \times 10^{-2})$
Sample size $n_i = 100$						
0	0.0109	0.05	$2.48 \times 10^{-2}(2.76 \times 10^{-2})$	0.0103	0.05	$1.43 \times 10^{-2}(1.26 \times 10^{-2})$
0.1	0.0511	0.1142	$1.47 \times 10^{-2}(1.83 \times 10^{-2})$	0.0469	0.1172	$1.07 \times 10^{-2}(1.35 \times 10^{-2})$
0.2	0.2559	0.349	$1.35 \times 10^{-2}(1.45 \times 10^{-2})$	0.2555	0.3744	$1.92 \times 10^{-2}(2.25 \times 10^{-2})$
Sample size $n_i = 500$						
0	0.0339	0.05	$1.51 \times 10^{-2}(1.47 \times 10^{-2})$	0.0329	0.05	$1.56 \times 10^{-2}(1.335 \times 10^{-2})$
0.1	0.3628	0.3774	$1.55 \times 10^{-2}(1.43 \times 10^{-2})$	0.377	0.3916	$1.59 \times 10^{-2}(1.42 \times 10^{-2})$
0.2	0.9349	0.9398	$1.56 \times 10^{-2}(1.21 \times 10^{-2})$	0.9602	0.9558	$1.72 \times 10^{-2}(2.07 \times 10^{-2})$

Table 3. Type I error and power of the test of equality of quantiles for sample sizes $n_i = 50, 100, 500$, at $p = 0.9$.

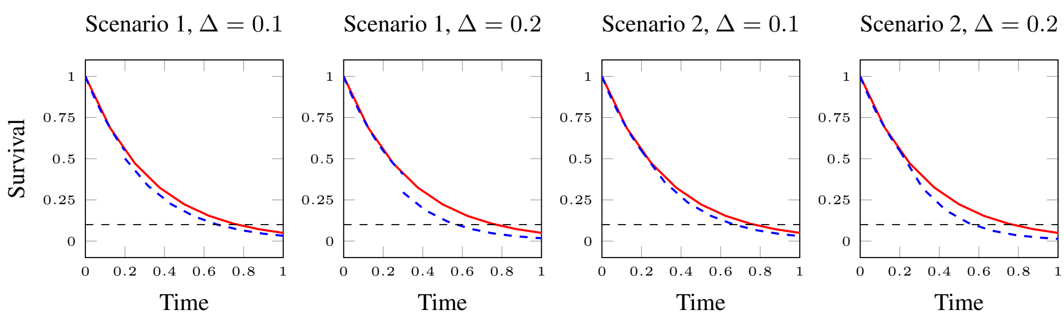

Figure 3. True survival curves for the test at $p = 0.05$ with 5% of censoring.

3.4 Test at $p = 0.25$, 25% of censoring

We illustrate the test at a low quantile. We set $\lambda_a = 0.1$, $\lambda_{\text{cens}} = 0.03$, $t_{\text{cut}} = 0.2$.

Δ	Scenario 1			Scenario 2		
	Empirical	**Formula**	**Comp. time**	**Empirical**	**Formula**	**Comp. time**
			Sample size $n_i = 50$			
0	0.0145	0.05	$8.84 \times 10^{-3}(9.96 \times 10^{-3})$	0.0198	0.05	$1.04 \times 10^{-2}(9.62 \times 10^{-3})$
0.1	0.0173	0.0502	$9.94 \times 10^{-3}(9.63 \times 10^{-3})$	0.0190	0.0504	$1.14 \times 10^{-2}(1.11 \times 10^{-2})$
0.2	0.0158	0.0508	$8.46 \times 10^{-3}(8.52 \times 10^{-3})$	0.0188	0.0171	$1.38 \times 10^{-2}(1.01 \times 10^{-2})$
			Sample size $n_i = 100$			
0	0.0304	0.05	$9.5 \times 10^{-3}(9.25 \times 10^{-3})$	0.0331	0.05	$1.38 \times 10^{-2}(1.00 \times 10^{-2})$
0.1	0.0287	0.0504	$9.4 \times 10^{-3}(8.74 \times 10^{-3})$	0.0383	0.0508	$1.14 \times 10^{-2}(1.16 \times 10^{-2})$
0.2	0.0349	0.0516	$1.06 \times 10^{-2}(9.30 \times 10^{-3})$	0.0377	0.0534	$1.27 \times 10^{-2}(9.65 \times 10^{-3})$
			Sample size $n_i = 500$			
0	0.0466	0.05	$1.17 \times 10^{-2}(1.22 \times 10^{-2})$	0.0475	0.05	$1.40 \times 10^{-2}(1.12 \times 10^{-2})$
0.1	0.0484	0.0520	$1.28 \times 10^{-2}(7.92 \times 10^{-3})$	0.0550	0.0540	$1.57 \times 10^{-2}(9.97 \times 10^{-3})$
0.2	0.0526	0.0582	$1.41 \times 10^{-2}(7.93 \times 10^{-3})$	0.0780	0.0673	$1.16 \times 10^{-2}(9.38 \times 10^{-3})$

Table 4. Type I error and power of the test of equality of quantiles for sample sizes $n_i = 50, 100, 500$, at $p = 0.25$.

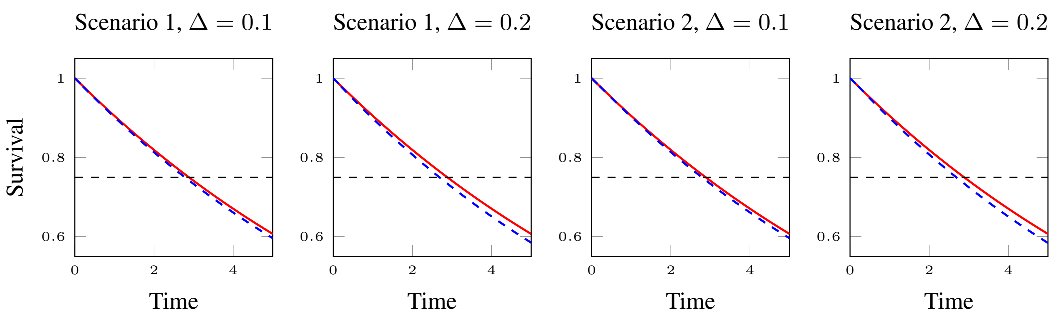

Figure 4. True survival curves for the test at $p = 0.05$ with 25% of censoring.

3.5 *Test at* $p = 0.05$*, 25% of censoring*

We show simulations for a low extreme quantile of survival. In order to do so, we let $\lambda_a = 0.1$, $\lambda_{\text{cens}} = 0.03$, $t_{\text{cut}} = 0.2$.

Δ	Scenario 1			Scenario 2		
	Empirical	**Formula**	**Comp. time**	**Empirical**	**Formula**	**Comp. time**
			Sample size $n_i = 50$			
0	0.0122	0.05	$1.55 \times 10^{-2}(1.149 \times 10^{-2})$	0.0117	0.05	$1.24 \times 10^{-2}(1.31 \times 10^{-2})$
0.1	0.0156	0.0565	$1.5 \times 10^{-2}(1.487 \times 10^{-2})$	0.0153	0.0512	$1.01 \times 10^{-2}(1.184 \times 10^{-2})$
0.2	0.0201	0.0820	$1.59 \times 10^{-2}(1.949 \times 10^{-2})$	0.0240	0.0627	$1.41 \times 10^{-2}(1.752 \times 10^{-2})$
			Sample size $n_i = 100$			
0	0.0281	0.05	$1.55 \times 10^{-2}(1.076 \times 10^{-2})$	0.0254	0.05	$1.3 \times 10^{-2}(1.84 \times 10^{-2})$
0.1	0.0356	0.0632	$1.58 \times 10^{-2}(1.199 \times 10^{-2})$	0.0281	0.0524	$1.04 \times 10^{-2}(1.994 \times 10^{-2})$
0.2	0.0613	0.11499	$1.37 \times 10^{-2}(1.169 \times 10^{-2})$	0.0517	0.0757	$1.2 \times 10^{-2}(1.39 \times 10^{-2})$
			Sample size $n_i = 500$			
0	0.0491	0.05	$1.7 \times 10^{-2}(1.82 \times 10^{-2})$	0.049	0.05	$1.46 \times 10^{-2}(1.36 \times 10^{-2})$
0.1	0.1073	0.1176	$1.4 \times 10^{-2}(1.23 \times 10^{-2})$	0.1103	0.1264	$1.36 \times 10^{-2}(1.466 \times 10^{-2})$
0.2	0.3771	0.3816	$1.37 \times 10^{-2}(1.3 \times 10^{-2})$	0.423	0.4470	$1.23 \times 10^{-2}(1.118 \times 10^{-2})$

Table 5. Type I error and power of the test of equality of quantiles for sample sizes $n_i = 50, 100, 500$, at $p = 0.05$.

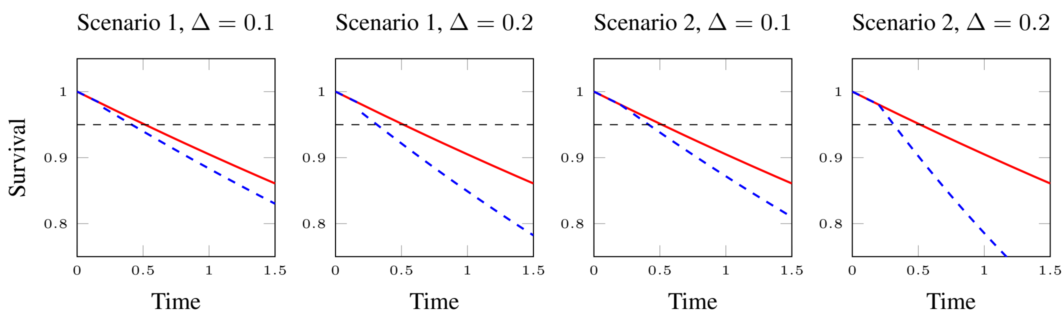

Figure 5. True survival curves. First and second columns correspond to Scenario 1, third and fourth columns correspond to Scenario 2.

3.6 Test at $p = 0.5$, 50% of censoring

We modify scenario 3.1, now with a higher censoring rate.

Δ	Scenario 1			Scenario 2		
	Empirical	**Formula**	**Comp. time**	**Empirical**	**Formula**	**Comp. time**
			Sample size $n_i = 50$			
0	0.0043	0.05	$1.11 \times 10^{-2}(1.25 \times 10^{-2})$	0.0044	0.05	$8.54 \times 10^{-3}(8.79 \times 10^{-3})$
0.1	0.0229	0.1112	$1.17 \times 10^{-2}(1.15 \times 10^{-2})$	0.0084	0.1071	$9.04 \times 10^{-3}(9.86 \times 10^{-3})$
0.2	0.2251	0.3586	$9.48 \times 10^{-3}(9.60 \times 10^{-3})$	0.1650	0.4032	$9.22 \times 10^{-3}(1.06 \times 10^{-2})$
			Sample size $n_i = 100$			
0	0.0149	0.05	$1.18 \times 10^{-2}(9.64 \times 10^{-3})$	0.0167	0.05	$9.38 \times 10^{-3}(9.40 \times 10^{-3})$
0.1	0.1102	0.1748	$1.16 \times 10^{-2}(1.07 \times 10^{-2})$	0.1021	0.1663	$8.72 \times 10^{-3}(9.28 \times 10^{-3})$
0.2	0.6461	0.6178	$1.03 \times 10^{-2}(9.63 \times 10^{-3})$	0.8340	0.6790	$1.04 \times 10^{-2}(1.05 \times 10^{-2})$
			Sample size $n_i = 500$			
0	0.0470	0.05	$1.30 \times 10^{-2}(9.13 \times 10^{-3})$	0.0436	0.05	$1.34 \times 10^{-2}(9.89 \times 10^{-3})$
0.1	0.6296	0.6249	$1.24 \times 10^{-2}(9.27 \times 10^{-3})$	0.7010	0.5954	$1.36 \times 10^{-2}(9.89 \times 10^{-3})$
0.2	1.0000	1.0000	$1.46 \times 10^{-2}(1.00 \times 10^{-2})$	1.0000	1.0000	$1.30 \times 10^{-2}(9.94 \times 10^{-3})$

Table 6. Type I error and power of the test of equality of quantiles for sample sizes $n_i = 50, 100, 500$, at $p = 0.5$, with more censoring.

References

1. Farah B., Latouche A., Bouaziz O. A note on a resampling procedure for estimating the density at a given quantile. *arXiv preprint arXiv:2509.02207*. 2025.
2. Kosorok MR. Two-sample quantile tests under general conditions. *Biometrika*. 1999;86(4):909–21.
3. Fleming TR, Harrington DP. *Counting Processes and Survival Analysis*. New York: John Wiley & Sons; 1991.
4. Lin C, Zhang L, Zhou Y. Conditional quantile residual lifetime models for right censored data. *Lifetime Data Anal*. 2015;21:75–96.
5. Földes A, Rejtő L, Winter BB. Strong consistency properties of nonparametric estimators for randomly censored data, II: Estimation of density and failure rate. *Period Math Hungar*. 1981;12:15–29.
6. Diehl S, Stute W. Kernel density and hazard function estimation in the presence of censoring. *J Multivar Anal*. 1988;25(2):299–310.
7. Marron JS, Padgett WJ. Asymptotically optimal bandwidth selection for kernel density estimators from randomly right-censored samples. *Ann Stat*. 1987;15:1520–35.